Lecture 16

Recall: $X_w = \bigcup_{w' \leq w} C_w$

$e^n = \{v \in \mathbb{R}^n, \|v\| < 1\}$ $\quad D^n = \{v \in \mathbb{R}^n, \|v\| \leq 1\}$

CW complex is a collection of embeddings $e^{n_\alpha} \xrightarrow[\varphi_\alpha]{} X$

s.t. each extends to cts map $D^{n_\alpha} \xrightarrow[\overline{\varphi}_\alpha]{} X$ (not nec. embedding).

Last time: Schubert cell C_w realized as image of emb.

$$\prod_{\alpha \in \Phi_\alpha^+} U_\alpha \longrightarrow G/B$$

$$(u_\alpha) \longmapsto (\prod u_\alpha) \cdot w x_0$$

Take $\prod U_\alpha = \prod \mathbb{C} \cong \mathbb{R}^{2\ell(w)} \cong e^{2\ell(w)}$ Why does it extend?

Ex. $G = SL_2\mathbb{C}$ $\quad B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$ $\quad W = \{e, \underset{s_\alpha}{\overset{\shortparallel}{w_0}}\}$ $\quad G/B \cong \mathbb{CP}^1$

$C_e = x_0 = [1:0]$ $\quad \alpha = e_1 - e_2$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} B \leftrightarrow [a:c]$

$\Phi^+ = \{\alpha\}$

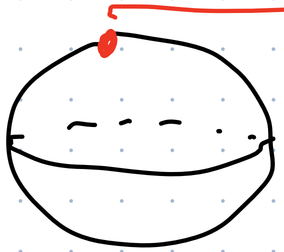
$\Phi_e^+ = \phi$ $\quad \Phi_{w_0}^+ = \{\alpha\}$

$U_\alpha = \left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}$ $\quad U_\alpha \longrightarrow G/B$

$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot \overset{w_0}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} B = \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix} B = [t:1]$

$C_{w_0} = \{[t:1]\}$

Thus



$C_e \cong \mathbb{C}^0 = pt.$

$C_{w_0} \cong \mathbb{C}$

General Bott-Samelson: $\prod U_\alpha \xrightarrow{\sim} C_w$ extends to

$$\underset{\prod \mathbb{CP}^1}{\overset{\prod U_\alpha}{\cap}} \longrightarrow X_w$$

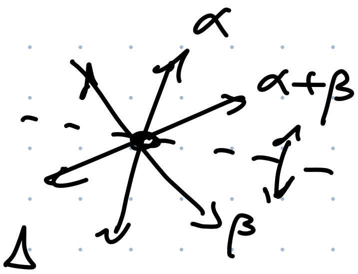
where $\partial \prod \mathbb{C}$ in $\mathbb{CP}^1$ maps to $\overline{C_w} - C_w$. $\quad$ Cor X_w rational

So the C_w giving a CW cplx structure is equivalent to saying that $(S^2)^n$ has a CW complex structure where $(S^2-pt)^n$ is the only $2n$-cell. Segue...

Now have: $\Phi, \Phi^+ \rightsquigarrow$ group W and $\ell : W \to \mathbb{Z}^{\geq 0}$.

$H_{2k}(G/B) \cong \mathbb{Z}^{b_{2k}} \qquad b_{2k} = \#\{w \in W \mid \ell(w) = k\}$

<u>Ex.</u> $\ell(s_\alpha) \geq 1$ since $s_\alpha(\alpha) = -\alpha$



Other $\beta \in \Delta$ $\quad s_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha \qquad \langle \alpha, \beta \rangle < 0$

So $s_\alpha(\beta) \in \beta + \mathbb{Z}^+\alpha$ for other $\beta \in \Delta$

$\Rightarrow s_\alpha(\beta) \in \Phi^+$ if $\beta \in \Phi^+$.

$\Rightarrow \ell(s_\alpha) = 1$.

In fact these are <u>all</u> and $H_2(G/B) \cong \mathbb{Z}^{rk(G)}$

Coxeter groups Recall $W \cong \langle s_\alpha \mid (s_\alpha s_\beta)^{n(\alpha,\beta)} \rangle$

where $n(\alpha, \beta) \in \{1, 2, 3, 4, 6\}$ $\qquad$ Combinatorics of Coxeter Groups GTM 231.

Fact (Björner-Brenti; for ex)

① $\ell(w) = \min\{k \mid \exists$ word of length k in s_α equal to $w\}$

② $w \leq w' \Leftrightarrow \exists$ word of length $\ell(w')$ for w' s.t. after crossing out one or more symbols, a word for w is obtained.

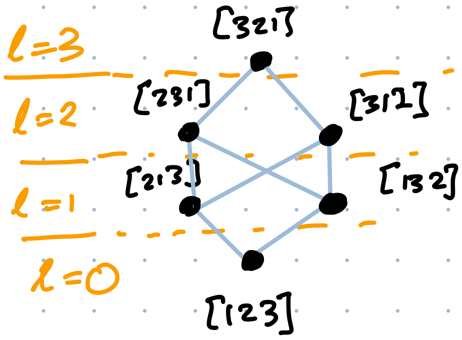
③ $\exists!$ element $w_0 \in W$ of maximal length, it is an invol, and mult by w_0 is an <u>anti-automorphism</u>.
$a < b \Leftrightarrow w_0 a > w_0 b$

Fact ③ also follows from G/B being a compact, oriented manifold as $H_{\dim G/B}(G/B) \cong \mathbb{Z}$. The element w_0 also has $w_0 \Phi^+ = \Phi^-$.

④ Chain property: $\forall\, v, w \quad v \leq w$, $\exists v = v_0, v_1, \dots, v_n = w$ $\ell(v_{i+1}) = \ell(v_i) + 1$.

Example $G = SL_3\mathbb{C}$ $\quad W = Sym_3 =$

[123]	[213]	[132]	[231]	[312]	[321]
e	a	b	ba	ab	$aba = bab$
	s_α	s_β			w_0
$\Phi^+_\sigma = \emptyset$	12	23	12 13	23 13	12 13 23



$\Phi^+ = \{e_i - e_j \mid i<j\}$ $\quad \sigma \in S_3$, then $\sigma \cdot e_i = e_{\sigma(i)}$

So $\Phi^+_\sigma = \{i,j \mid i<j \text{ and } \sigma(i) > \sigma(j)\}$ INVERSIONS

$C_{123} = \{x_0\}$ $\quad C_{213} \cong C_{132} \cong \mathbb{C}$ $\quad X_{213} \cong X_{132} \cong \mathbb{CP}^1$.

$G/B = $ flags in $\mathbb{C}^3 = \{(p,\ell) \mid p \text{ point in } \mathbb{CP}^2,\ \ell \text{ line in } \mathbb{CP}^2\}$

$(v_1 | v_2 | v_3) \cdot B \leftrightarrow [v_1] \in \mathbb{CP}^2,\ \overline{[v_1][v_2]} \subset \mathbb{CP}^2$
$= [\text{span } v_1, v_2]$

$$C_{213} = \underbrace{B \cdot}_{\text{"upward" row ops}} \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1\end{pmatrix} \cdot B = \begin{pmatrix} 1&*&*\\0&1&*\\0&0&1\end{pmatrix} \cdot \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1\end{pmatrix} B = \begin{pmatrix} *&1&0\\1&0&0\\0&0&1\end{pmatrix} B$$

"upward" row ops

$= \{(p,\ell) \mid \ell = [\text{span } e_1, e_2],\ p \neq [e_1]\}$

$X_{213} = \{(p,\ell) \mid \ell = \text{span}[e_1,e_2]\} \cong \mathbb{CP}^1 = \mathbb{P}(\text{span}(e_1,e_2))$

$$C_{132} = B \cdot \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0\end{pmatrix} B = \begin{pmatrix} 1&*&*\\0&1&*\\0&0&1\end{pmatrix} \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0\end{pmatrix} B = \underbrace{\begin{pmatrix} 1&*&*\\0&*&1\\0&1&0\end{pmatrix}}_{\text{result of upward row ops}} B = \underbrace{\begin{pmatrix} 1&0&0\\0&*&1\\0&1&0\end{pmatrix}}_{\text{result of rightward col ops}} B$$

result of upward row ops

result of rightward col ops.

$= \{(p,\ell) \mid p = [e_1],\ \ell \neq [e_1, e_2]\}$

$X_{132} = \{(p,\ell) \mid p = [e_1]\}$.

Similar analysis: $\quad X_{231} = $ closure of $\begin{pmatrix} *&1&0\\ *&0&1\\ 1&0&0\end{pmatrix} B = \{[e_1] \in \ell\}$

$X_{312} = \{p \in [e_1, e_2]\}$

$C_{321} = \text{span } v_1 e_2 e_3 = \text{span } v_1 v_2 e_3 = \mathbb{C}^3$ $\quad X_{321} = \text{Flag}(\mathbb{C}^3)$.

Note
$$\begin{tikzcd}[column sep=small] & \mathrm{Flag}(\mathbb{C}^3) = G/B \arrow[dl] \arrow[dr] & \\ G/P_\alpha & & G/P_\beta \\ (\mathbb{CP}^2)^* & & \mathbb{CP}^2 \end{tikzcd}$$

And both are $\mathbb{CP}^1$ bundles. The 1-dim Schubert vars are fibers of these.

So: $\mathrm{Flag}(\mathbb{C}^3) = \mathbb{CP}^1 \wedge \mathbb{CP}^1 \cup \mathbb{C}^2 \cup \mathbb{C}^2 \cup \mathbb{C}^3$

Exercise. Attaching map desc? $S^3 \longrightarrow \mathbb{CP}^1 \wedge \mathbb{CP}^1$?

$$H_k(\mathrm{Flag}(\mathbb{C}^3)) = \begin{cases} \mathbb{Z} & k = 0, 6 \\ \mathbb{Z}^2 & k = 2, 4 \\ 0 & \text{else} \end{cases}$$

Example $G = SL_n\mathbb{C}$. $W = \mathrm{Sym}_n$. Gen by $\alpha_1 \dots \alpha_{n-1}$
$\Phi^+ = \{e_i - e_j \mid i < j\}$ $\qquad \alpha_i = $ switch $i, i+1$.

$w_0 = (n \;\; n-1 \;\; n-2 \;\dots\; 1)$ $\quad \ell(w_0) = |\Phi^+| = \frac{n(n+1)}{2} = \dim_{\mathbb{C}} G/B = \dim_{\mathbb{C}} \mathrm{Flag}\,\mathbb{C}^n$.

$\sigma \in W$ view as map $[n] \to [n]$.
$\ell(\sigma) = |\Phi^+_\sigma| = \#\{i,j \mid i<j,\ \sigma(i) > \sigma(j)\}$.

The left action of B on a matrix doesn't change the dim of $F_i \cap E_j$ where $F_i = $ span cols $1 \dots i$ $\quad E_j = $ span $e_1 \dots e_j$

Let $d_{ij} = \dim(\sigma F_i) \wedge E_j = \#\, 1$ entries in the $j \times i$ upper left submatrix of σ
$= \#\{k \mid \sigma(k) \le j,\ k \le i\}$

Then $X_{ij} = \{F \mid \dim F_i \cap E_j \geq d_{ij}\}$ $\qquad C_{ij} = $ equality not $\geq$

e.g. $d_{ij} = n-i-j$ is "expected" and corresp to w_0, the open cell
$d_{ij} = \min(i,j)$ is when $F_i = E_i$. $\quad \sigma = e$ $\quad$ point!